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VARIATIONAL PROPERTIES OF FUNCTIONS OF THE MEAN CURVATURES FOR HYPERSURFACES IN SPACE FORMS

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Dedicated to the author's father on his 70th birthday

Introduction

The formula for the first variation, with fixed boundary, of the volume integral for a hypersurface M^n in Euclidean $(n+1)$-space E^{n+1} is well-known. It is, [4, p. 178], $\delta \int_M 1 \cdot dV = -n \int_M \sigma_1(N \cdot \xi) dV$, where σ_1 is the first mean curvature, N is the unit normal, and ξ is the deformation vector. Recently this classical formula has been generalized by several mathematicians including Pinl and Trapp [10] and the author [12]. In [12] we show that if σ_r, $r = 0, 1, \cdots, n$, denotes the r-th mean curvature function, then $\delta \int_M \sigma_r dV = -(n-r) \cdot \int_M \sigma_{r+1}(N, \xi) dV$. This is shown in [10] when $r = 1$ or n. We prove similar formulas in [12] for submanifolds of arbitrary codimension when r is even. The results of [10] and [12] for hypersurfaces are proved by Rund [13] in a more general setting. The object of the present paper is to study the variation of $\int_M f(S_1, \cdots, S_n) dV$, where M is a hypersurface in a space form $N^{n+1}(c)$ of curvature c, $S_r = C_r^n \sigma_r$ is the r-th elementary symmetric function of the principal curvatures (C_r^n being the binomial coefficient), and f is any smooth function. If $c = 0$, we also consider $\int_M f(S_1, \cdots, S_n, P, Q) dV$, where P is the support function, and $2Q$ is the square of the length of the position vector. Many of our results could be derived from the theory in [13] but it appears that because we study a less general case here our methods are more elementary than those of [13].

We begin by deriving the formula for the first variation of our integral as well as the formula for the second variation in those cases (see above) studied

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by Rund and the author (§ 1). In a study of the Euclidean case we derive the differential equations which characterize extremals of $\int_M S_r dV$, we prove that convex hypersurfaces with vanishing gauss-kronecker curvature are semi-stable extremals of $\int_M S_{n-1} dV$, and we consider an integral $\int_M S_r^{n/r} dV$ discussed by Chen [2] (§ 2). Next we consider similar questions for hypersurfaces in the unit sphere (§ 3). We continue with a discussion on the nature of integral formulas, and illustrate our ideas with a new derivation of the well-known Minkowski-Hsiung integral formulas (§ 4). We conclude with a potpourri of observations and questions (§ 5).

In the present paper everything in sight is of class C^∞, and all manifolds are compact, possibly with boundary, and orientable. The letters h, i, j, k, l, when used as indices (with or without subscripts), are used in the sense of classical tensor analysis. In particular, the summation convention is in effect for these indices (all sums going from 1 to n) and we raise and lower them with the usual abandon.

1. The fundamental formulas

We begin with some algebra. Let V be a (real) n-dimensional vector space, and $B: V \to V$ be a diagonalizable linear transformation (i.e., V has a basis of eigenvectors of B). We fix a basis $v_1, \cdots, v_n$ of V, and denote the matrix of B relative to this basis by (b_i^j). For $r = 0, 1, \cdots, n$ let S_r denote the r-th elementary symmetric function of the eigenvalues $k_1, \cdots, k_n$ of B. Thus $S_0 = 1$, $S_1 = k_1 + \cdots + k_n, \cdots, S_n = k_1 \cdots k_n$.

Remark. We are really only interested in the situation where V is a tangent space of a hypersurface, and B is the shape operator, i.e., B is the symmetric linear transformation associated by the metric with the second fundamental form. In this case the S_r are the modified mean curvatures (see Introduction).

We can express $S_0, \cdots, S_n$ directly in terms of B.

Definition. The r-th Newton transformation (or tensor), $r = 0, 1, \cdots, n$, is the linear transformation $T_r = S_r I - S_{r-1} B + \cdots + (-1)^r B^r$.

Remarks. 1. If we denote the matrix, relative to $v_1, \cdots, v_n$, of B^q by $(b^{(q)}{}_i{}^j)$, then the matrix of T_r is

$$T_{ri}{}^j = S_r \delta_i{}^j - S_{r-1} b_i{}^j + S_{r-2} b^{(2)}{}_i{}^j + \cdots + (-1)^r b^{(r)}{}_i{}^j .$$

2. The Newton transformations can be defined inductively by $T_0 = I$, $T_{r+1} = S_{r+1} I - B T_r$.

3. Since T_r is a polynomial in B, it is clear that $T_r B = B T_r$, and that T_r has the same eigenvectors as B.

4. By direct computation or use of the Hamilton-Cayley theorem of linear algebra, one sees that $T_n = 0$.

The following equation is called Newton's formula [14, p. 81], which justifies the name we have given the transformations T_r:

$$(1) \qquad (r + 1)S_{r+1} = \text{Trace } (BT_r) .$$

One shows that (1) is equivalent to the classical Newton's formula by computing the right hand side of (1) relative to a basis of eigenvectors of B.

The next lemma is crucial to our discussion.

Lemme A. *Let $B = B(t)$ be a smooth one-parameter family of diagonalizable linear transformations of V. Then for $r = 0, 1, \cdots, n$ we have*

$$(2) \qquad \partial S_{r+1}/\partial t = \text{Trace } (\partial B/\partial t \cdot T_r) .$$

We have already essentially proved this fact [11, Lemma A], but that proof was rather cumbersome. We get a neater proof here by using a second representation of T_r.

We recall the definition of the generalized Kronecker symbols. If $i_1, \cdots, i_q$ and $j_1, \cdots, j_q$ are integers between 1 and n, then $\varepsilon_{i_1 \cdots i_q}{}^{j_1 \cdots j_q}$ is $+1$ or -1 according as the i's are distinct and the j's are an even or odd premutation of the i's, and is 0 in all other cases.

Proposition A. *The matrix of T_r is given by*

$$(3) \qquad T_{ri}{}^{j} = (r!)^{-1} \varepsilon_{i_1 \cdots i_r i}{}^{j_1 \cdots j_r j} b_{j_1}{}^{i_1} \cdots b_{j_r}{}^{i_r} .$$

One verifies (3) by computing both sides of the equation in the case where $v_1, \cdots, v_n$ is a basis of eigenvectors for B.

We can now prove Lemma A by differentiating both sides of (1) with respect to t. (2) follows by using (3) in the right hand side of (1) and by observing the symmetry properties of the Kronecker symbols.

Now let us consider a one-parameter family $X = X_t : M^n \to N^{n+1}(c)$ of immersions of the n-manifold M into the space form $N^{n+1}(c)$ of curvature c. The family X induces in each tangent space of M a one-parameter family $B = B(t)$ of diagonalizable linear transformations, namely, the shape operators for each immersion, as well as a family $dV = dV_t$ of volume elements. Denote the deformation vector field $\partial X/\partial t$ and the unit normal field in $N^{n+1}(c)$ by ξ and N, respectively, and set $\lambda = \langle \xi, N \rangle$ where $\langle \, , \rangle$ denotes the metric on $N^{n+1}(c)$. In this paper we consider only deformations which leave ∂M strongly fixed in the sense that both λ and its gradient vanish on ∂M, and the tangential component of ξ is parallel to $X(\partial M)$ along ∂M. Of course, if M is compact and ∂M is empty, then there is no restriction on the deformation. We can now state our main theorem.

Theorem A. *Suppose that f is any smooth function of n variables. Then*

$$\frac{d}{dt}\int_M f(S_1,\cdots,S_n)dV$$

$$(4)\quad \begin{aligned} &= \int_M \lambda\Big\{-S_1 f(S_1,\cdots,S_n) + \sum_{r=1}^{n}(S_rS_1-(r+1)S_{r+1})D_rf(S_1,\cdots,S_n) \\ &\quad + \sum_{r=1}^{n}(D_rf(S_1,\cdots,S_n))_{,ij}T_{r-1}{}^{ij} \\ &\quad + \sum_{r=1}^{n}D_rf(S_1,\cdots,S_n)c(n-r+1)S_{r-1}\Big\}dV\,. \end{aligned}$$

Moreover, if $P=\langle X,N\rangle$ *is the support function,* $c=0$, *and* $Q=\frac{1}{2}|X|^2$, *then for any smooth function* f *of* $n+2$ *variables we have*

$$\frac{d}{dt}\int_M f(S_1,\cdots,S_n,P,Q)dV$$

$$(5)\quad \begin{aligned} &= \int_M \lambda\Big\{-S_1 f(S_1,\cdots,Q) + \sum_{r=1}^{n}(S_rS_1-(r+1)S_{r+1})D_rf(S_1,\cdots,Q) \\ &\quad + \sum_{r=1}^{n}(D_rf(S_1,\cdots,Q))_{,ij}T_{r-1}{}^{ij} \\ &\quad + (PS_1+(n+1))D_{n+1}f(S_1,\cdots,Q) \\ &\quad + (D_{n+1}f(S_1,\cdots,Q))_{,j}Q_{,}{}^{j} + PD_{n+2}f(S_1,\cdots,Q)\Big\}dV\,. \end{aligned}$$

Here the commas, as in $D_rf,_{ij}$, *indicate covariant differentiation relative to the metric induced by* X, *and the summation convention is in effect (see the end of Introduction).*

Proof. We will verify only (5); the proof of (4) is entirely similar. Our calculations will be in terms of local coordinates $x^1,\cdots,x^n$ on M. Then we let g_{ij}, $b_i{}^j$ and μ^j represent, respectively, the metric tensor, the shape operator and the tangential component of ξ. By the chain rule we have

$$(6)\quad \begin{aligned} \frac{d}{dt}\int_M f(S_1,\cdots,S_n,P,Q)dV &= \int_M\Big\{\sum_{r=1}^{n}D_rf(S,\cdots,Q)\frac{\partial S_r}{\partial t} \\ &\quad + D_{n+1}f(S_1,\cdots,Q)\frac{\partial P}{\partial t} + D_{n+2}f(S_1,\cdots,Q)\frac{\partial Q}{\partial t}\Big\}dV \\ &\quad + \int_M f(S_1,\cdots,Q)\frac{\partial}{\partial t}(dV)\,. \end{aligned}$$

It is a standard fact that $(\partial/\partial t)(dV)=(-S_1\lambda+\mu^j{}_{,j})dV$, proved by observing that $dV=(\det(g_{ij}))^{1/2}dx^1\cdots dx^n$ and computing $(\partial/\partial t)(\det(g_{ij}))$. Thus to prove (6) we must compute $\partial S_r/\partial t$, $\partial P/\partial t$, $\partial Q/\partial t$.

Lemma A implies that $\partial S_r/\partial t = T_{r-1i}{}^j \partial b_j{}^i/\partial t$. However $b_j{}^i = g^{ik}b_{jk}$, where $b_{jk} = -\langle \partial X/\partial x^j, \partial N/\partial x^k \rangle$. Thus to compute $\partial b_j{}^i/\partial t$ we must compute $\partial g^{ik}/\partial t$, $\langle \partial^2 X/\partial t \partial x^j, \partial N/\partial x^k \rangle$ and $\langle \partial X/\partial x^j, \partial^2 N/\partial t \partial x^k \rangle$. Since the calculations are both elementary and boring, let us just indicate how they go. In all the calculations we use the fact that $\partial^2/\partial t \partial x^k = \partial^2/\partial x^k \partial t$, together with $\xi = \partial X/\partial t$. Indeed, it was just to get the ease of manipulation which the commutivity of differentiation allows which caused us to use local coordinates instead of, say, moving frames. We compute $\partial g^{ik}/\partial t$ by differentiating both sides of the equation $g^{ik}g_{kj} = \delta_j{}^i$ with respect to t (recall that $g_{kj} = \langle \partial X/\partial x^k, \partial X/\partial x^j \rangle$). We get $\partial g^{ik}/\partial t = g^{ih}g^{jk}(2\lambda b_{hj} - (\mu_{h,j} + \mu_{j,h}))$. The rest of the computation requires the use of

$$\text{(7)} \qquad \partial N/\partial t = -g^{ij}(\lambda_{,i} + \mu^k b_{ki})\partial X/\partial x^j \ .$$

This equation is proved by differentiating both sides of $N \cdot \partial X/\partial x^k = 0$ with respect to t. After a bit of index manipulations we eventually arrive at

$$\text{(8)} \qquad \partial b_j{}^i/\partial t = E_j{}^i - F_j{}^i + H_j{}^i + I_j{}^i + K_j{}^i \ ,$$

where $E_j{}^i = \lambda g^{ih}b_{jk}b_h{}^k = \lambda b^{(2)}{}_j{}^i$, $F_j{}^i = g^{ih}g^{kl}b_{jl}\mu_{h,k}$, $H_j{}^i = g^{ih}b_h{}^k\mu_{k,j}$, $I_j{}^i = g^{ih}g_{kj}\lambda_{,}{}^k{}_h$ and $K_j{}^i = g^{il}g_{hj}\mu^k b_k{}^h{}_{,l}$. In order to compute $\partial S_{r+1}/\partial t$ we must multiply both sides of (8) by $T_{ri}{}^j$ and sum over i and j. Now $-F_j{}^iT_{ri}{}^j + H_j{}^iT_{ri}{}^j = 0$ because $T_rB = BT_r$. Also Lemma A and the Codazzi equations imply that $K_j{}^iT_{ri}{}^j = S_{r+1,j}\mu^j$. From our original definition of the Newton tensors and (1) we see that $E_j{}^iT_{ri}{}^j = \lambda(S_1S_{r+1} - (r+2)S_{r+2})$. Finally, it is clear that $I_j{}^iT_{ri}{}^j = T_r{}^{ij}\lambda_{,ij}$. Thus we have proved that

$$\text{(9)} \qquad \partial S_{r+1}/\partial t = \lambda(S_1S_{r+1} - (r+2)S_{r+2}) + T_r{}^{ij}\lambda_{,ij} + S_{r+1,j}\mu^j \ .$$

Remark. The analogous formula in $N^{n+1}(c)$, c arbitrary, is

$$\text{(9c)} \qquad \begin{aligned} \partial S_{r+1}/\partial t = {} & \lambda(S_1S_{r+1} - (r+2)S_{r+2}) \\ & + T_r{}^{ij}\lambda_{,ij} + S_{r+1,j}\mu^j + c\lambda(n-r)S_r \ . \end{aligned}$$

Equation (7), together with the obvious fact that $Q_{,j} = (X, \partial X/\partial x^j)$, yields the formula for $\partial P/\partial t = (\xi, N) + (X, \partial N/\partial t)$, i.e., $\partial P/\partial t = \lambda - \lambda_{,j}Q_{,}{}^j - \mu^j b_j{}^k Q_{,k}$.

Finally one easily computes that $\partial Q/\partial t = (\xi, X) = \mu^j Q_{,j} + \lambda P$. By combining all these formulas we get

$$\text{(10)} \qquad \begin{aligned} \frac{d}{dt}\int_M f(S_1, \cdots, Q)dV = \int_M \Big\{ & \sum_{r=1}^n D_r f(S_1, \cdots, Q)(\lambda(S_rS_1 - (r+1)S_{r+1}) \\ & + T_{r-1}{}^{ij}\lambda_{,ij} + S_{r,j}\mu^j) + D_{n+1}f(S_1, \cdots, Q)(\lambda - \lambda_{,j}Q_{,}{}^j - \mu^j b_j{}^k Q_{,k}) \\ & + D_{n+2}f(S_1, \cdots, Q)\mu^j Q_{,j} + f(S_1, \cdots, Q)(-S_1\lambda + \mu^j{}_{,j})\Big\} dV \ . \end{aligned}$$

Since $P_{,j} = -b_j{}^k Q_{,k}$, it is clear that

$$\sum_{r=1}^{n} D_r f(S_1, \cdots, Q) S_{r,j}\mu^j - D_{n+1} f(S_1, \cdots, Q) b_j{}^k Q_{,k}\mu^j$$
$$+ D_{n+2} f(S_1, \cdots, Q) Q_{,j}\mu^j + f(S_1, \cdots, Q)\mu^j{}_{,j} = (f(S_1, \cdots, Q)\mu^j)_{,j} \, .$$

Now Stokes' theorem tells us that $\int_M (f(S_1, \cdots, Q)\mu^j)_{,j} dV = \int_{\partial M} f(S_1, \cdots, Q) \cdot (\mu^j \gamma_j) dA$, where γ_j are the covariant components of the unit normal to ∂M in M, and dA is the volume element on ∂M. Since, by hypothesis, $\mu^j \gamma_j = 0$ on ∂M, we see that the terms involving μ^j in (10) do not contribute anything to the answer.

Similarly, if we use integration by parts to get rid of the derivatives of λ, we see that $\int_M D_r f(S_1, \cdots, Q) T_{r-1}{}^{ij} \lambda_{,ij} dV = \int_M (D_r f(S_1, \cdots, Q) T_{r-1}{}^{ij})_{,ij} \lambda dV$ and $\int_M -D_{n+1} f(S_1, \cdots, Q) \lambda_{,j} Q_{,}{}^j dV = \int_M (D_{n+1} f(S_1, \cdots, Q) Q_{,}{}^j)_{,j} \lambda dV$. The boundary terms one would expect after integration by parts vanish because of our hypothesis that λ and $\lambda_{,j} = 0$ on ∂M.

We get precisely (5) after we observe that $(D_r f(S_1, \cdots, Q) T_{r-1}{}^{ij})_{,ij} = (D_r f(S_1, \cdots, Q))_{,ij} T_{r-1}{}^{ij}$ and that $Q_{,}{}^j{}_j = n + S_1 P$. This last statement is obvious, while the first is true because as we proved in [11, Lemma B] the Newton tensors enjoy the following property.

Proposition B. *The Newton tensors are divergence-free, i.e.,* $T_r{}^{ij}{}_{,j} = 0$.

The existence of covariant derivatives in (4) and (5) interferes with the computation of the second variation. Fortunately these derivatives vanish in the most important cases.

Theorem B. *Suppose that X and M are as in Theorem A. Then*

$$\text{(a)} \qquad \frac{d}{dt}\int_M S_r dV = \int_M \lambda\{-(r+1)S_{r+1} + c(n-r+1)S_{r-1}\} dV .$$

If, in addition, the immersion $X_t|_{t=0}$ yields an extremal value, i.e., $-(r+1)S_{r+1} + c(n-r+1)S_{r-1} = 0$ *when* $t = 0$, *then*

$$\text{(b)} \qquad \begin{aligned} \frac{d^2}{dt^2}\int_M S_r dV|_{t=0} = \int_M \{&\lambda^2[(r+1)(r+2)S_{r+2} \\ &- c(r(n-r+1) + (r+1)(n-r))S_r \\ &+ c^2(n-r+1)(n-r+2)S_{r-2}] \\ &+ ((r+1)T_r{}^{ij} - c(n-r+1)T_{r-2}{}^{ij})\lambda_{,i}\lambda_{,j}\} dV \, . \end{aligned}$$

Proof. For (a) use (4) with $f = S_r$. For (b) we first have

$$\frac{d^2}{dt^2}\int_M S_r dV|_{t=0} = \frac{d}{dt}\int_M (-(r+1)S_{r+1} + c(n-r+1)S_{r-1})\lambda dV|_{t=0}$$
$$= \int_M \left\{\frac{\partial}{\partial t}(-(r+1)S_{r+1} + c(n-r+1)S_{r-1})\right\}\lambda dV|_{t=0}$$
$$+ \int_M (-(r+1)S_{r+1} + c(n-r+1)S_{r-1})\frac{\partial}{\partial t}(\lambda dV)|_{t=0}\,.$$

Since by hypothesis the coefficient of $(\partial/\partial t)(\lambda dV)$ vanishes when $t = 0$, the last integral vanishes. Then (b) follows by applying (9) to the remaining integral.

2. Hypersurfaces in Euclidean space

Consider the variational problem $\delta \int_M S_r dV = 0$. Theorem B (a) with $c = 0$, gives that the Euler-Lagrange equation is $S_{r+1} = 0$.

Definition. A hypersurface in Euclidean space is said to be r-minimal if S_{r+1} vanishes identically.

Remark. The classical minimal hypersurfaces are the 0-minimal ones.

It is important to express r-minimality in terms of differential equations. Let $X = (X_1, \cdots, X_{n+1})$ be the position vector for a hypersurface M in E^{n+1}, and let $X_{,ij}$ denote the $(n+1)$-tuple $(X_{1,ij}, \cdots, X_{n+1,ij})$, where $\phi_{,ij}$ denotes the second covariant derivative for a real-valued function ϕ on M. It is easy to show that $X_{,ij} = b_{ij}N$. Thus Newton's formulas imply that $T_r{}^{ij}X_{,ij} = T_r{}^{ij}b_{ij}N = (r+1)S_{r+1}N$, and we have proved

Theorem C. *A hypersurface in Euclidean space is r-minimal if and only if each component of its position vector satisfies the partial differential equation* $T_r{}^{ij}\phi_{,ij} = 0$, *which can be put into divergence form* $(T_r{}^{ij}\phi_{,i})_{,j} = 0$ *since the Newton tensors are divergence-free.*

Remarks. 1. There exists no closed r-minimal hypersurface in E^{n+1} for $r < n$ since it is well-known that such a hypersurface has convex points at which $S_{r+1} \neq 0$.

2. Although it is not completely relevant to this paper, it is of interest to derive a system of partial differential equations which characterize the constancy of a given S_r. Let $N = (N_1, \cdots, N_{n+1})$ be the unit normal to a hypersurface M. It is easy to show that if A is any vector in E^{n+1} and $\phi = \langle N, A\rangle : M \to R$, then $\phi_{,ij} = -b^{(2)}{}_{ij}\phi - b_i{}^k{}_{,j}\langle \partial X/\partial x^k, A\rangle$. Thus by Newton's formulas, Codazzi's equations and Lemma A we obtain $T_r{}^{ij}\phi_{,ij} = -(S_1S_{r+1} - (r+2)S_{r+2})\phi - S_{r+1,k}\langle \partial X/\partial x^k, A\rangle$, and have proved

Proposition C. *The* $(r+1)$*-st mean curvature of a hypersurface in space is constant if and only if each component of the unit normal satisfies the partial differential equation* $T_r{}^{ij}\phi_{,ij} = -(S_1S_{r+1} - (r+2)S_{r+2})\phi$ *which can be put into divergence form as in Theorem C.*

A number of authors, e.g., [3], [6] have considered hypersurfaces in E^{n+1} for which $S_n = 0$. In our terminology they have been studying $(n-1)$-minimal hypersurfaces. If $n = 2$ these are just the developable surfaces in E^3. Because $S_{n+1} = 0$ by definition, the second variation for an $(n-1)$-minimal hypersurface is particularly simple. It is (cf. Theorem $B(b)$ with $c = 0$) $\delta^2 \int_M S_{n-1} dV = \int_M n T_{n-1}{}^{ij} \lambda_{,i} \lambda_{,j} dV$. Now it is easy to show that the eigenvalues of $T_{n-1}{}^{ij}$ are $k_2 k_3 \cdots k_n, k_1 k_3 \cdots k_n, \cdots, k_1 k_2 \cdots k_{n-1}$, If the hypersurface happens to be locally convex, then $k_q \geq 0$, $q = 1, \cdots, n$. Thus the following result is obvious.

Proposition D. *A convex hypersurface in E^{n+1} with identically vanishing Gauss-Kronecker curvature S_n is a semi-stable solution of the variational problem $\delta \int_M S_{n-1} dV = 0$ in the sense that the second variation is semi-definite.*

We now consider the problem, suggested by Chen [2], of minimizing $\int_M S_r^{n/r} dV$. If either r divides n or S_r is always positive, we can be sure that the integrand is smooth. With either of these smoothness conditions we conclude, after applying (5),

Theorem D. *The Euler-Lagrange equation for $\delta \int_M S_r^{n/r} dV = 0$ is* $(n S_r^{(n-r)/r})_{,ij} T_{r-1}{}^{ij} = -S_r^{(n-r)/r}((n-r) S_1 S_r - n(r+1) S_{r+1})$.

Corollary. *Any closed strictly convex hypersurface which is an extremal of the problem $\delta \int_M S_r^{n/r} dV = 0$ must be a hypersphere.*

Proof. Any extremal must be a solution of the above differential equation, the left hand side of which is a divergence since T_{r-1} is divergence-free, so by Stokes' theorem we get

$$-\int_M S_r^{(n-r)/r}((n-r) S_1 S_r - n(r+1) S_{r+1}) = \int_M n(S_r^{(n-r)/r}{}_{,j} T_{r-1}{}^{ij})_{,j} dV = 0 .$$

For a strictly convex hypersurface, $S_r > 0$ and $(n-r) S_1 S_r - n(r+1) S_{r+1} \geq 0$, the last inequality becoming an equation only at umbilics [5]. The vanishing of the above integrals, when combined with convexity, implies that $(n-r) S_1 S_r - n(r+1) S_{r+1} \equiv 0$. Thus the hypersurface is totally umbilic, i.e., is a hypersphere.

3. Hypersurfaces in the unit sphere

We begin with the variational problem $\delta \int_M S_r dV = 0$ for a hypersurface in the unit sphere $S^{n+1} \subset E^{n+2}$. By Theorem B we see that the Euler-Lagrange equation is $-(r+1)S_{r+1} + c(n-r+1)S_{r-1} = 0$ with $c = 1$.

Definition. A hypersurface in a sphere of curvature c is said to be r-minimal if $-(r+1)S_{r+1} + c(n-r+1)S_{r-1} = 0$.

As in § 2 one easily computes $X_{,ij} = (X_{1,ij}, \cdots, X_{n+2,ij})$; the result is

$$X_{,ij} = b_{ij}N - g_{ij}X . \tag{11}$$

Suppose that $r \neq 1$. If we multiply both sides of (11) by $(r-1)T_r{}^{ij} - (n-r+1)T_{r-2}{}^{ij}$ and sum over i and j, we get

$$\begin{aligned} &((r-1)T_r{}^{ij} - (n-r+1)T_{r-2}{}^{ij})X_{,ij} \\ &\quad = ((r-1)(r+1)S_{r+1} - (n-r+1)(r-1)S_{r-1})N \\ &\quad\quad - ((r-1)(n-r)S_r - (n-r+1)(n-r+2)S_{r-2})X . \end{aligned}$$

As an immediate consequence we have

Theorem E. *A hypersurface in S^{n+1} is r-minimal, $r \neq 1$, if and only if each component of the position vector satisfies the partial differential equation*

$$\begin{aligned} &((r-1)T_r{}^{ij} - (n-r+1)T_{r-2}{}^{ij})\phi_{,ij} \\ &\quad = -((r-1)(n-r)S_r - (n-r+1)(n-r+2)S_{r-2})\phi . \end{aligned}$$

Remarks. 1. The above equation does not characterize 1-minimal hypersurfaces. The equation for 1-minimality is $2S_2 = n$, and an alternate one is $T_1{}^{ij}X_{,ij} = nN - (n-1)S_1X$ which can be obtained by multiplying (11) by $T_1{}^{ij}$.

2. As in Proposition B we can characterize those hypersurfaces in S^{n+1} with S_{r+1} = constant, but we merely state the result as follows:

Proposition D. *Let M be a hypersurface in S^{n+1} with position vector $X = (X_1, \cdots, X_{n+2})$ and unit normal $N = (N_1, \cdots, N_{n+2})$. A necessary and sufficient condition that S_{r+1} be constant is that each vector $A \in E^{n+2}$ satisfy the equation*

$$T_r{}^{ij}\langle N, A\rangle_{,ij} = -(S_1S_{r+1} - (r+2)S_{r+2})\langle N, A\rangle + (r+1)S_{r+1}\langle X, A\rangle .$$

In particular, hypersurfaces with $S_{r+1} = 0$ are characterized by the fact that each component of the unit normal satisfies the equation

$$T_r{}^{ij}\phi_{,ij} = (r+2)S_{r+2}\phi .$$

Examples. 1. The totally geodesic equators and the small hyperspheres of radius $((n-r)/r)^{1/2}$ are r-minimal in S^{n+1} with the following two exceptions. $r = 1$: the equators are not 1-minimal; $r = n$: the small hyperspheres are not n-minimal.

2. If M is the Hsiang immersion [7] of $SO(3)/(z_2 + z_2)$ into S^4, then $S_1 = S_3 = 0$, so M is 0-minimal and 2-minimal.

3. No 0-minimal hypersurface is also 1-minimal, for if $S_1 = 0$ then $S_2 \leq 0$, while the condition for 1-minimality is $2S_2 = n > 0$.

4. The only closed 1-minimal surfaces in S^3 are the small spheres of radius $1/\sqrt{2}$. Indeed, the condition for 1-minimality is $2S_2 = 2$, or $S_2 = 1$. But it is easy to show that for a surface in S^3, $S_2 = K - 1$, where K is the gauss curvature. Thus $K = 2$, so the small spheres of radius $1/\sqrt{2}$ are 1-minimal. On the other hand, closed surfaces in S^3 with gauss curvature ≥ 1 at all points are rigid [1].

5. If the immersion $X: M \to S^{n+1}$ has nondegenerate gauss map, i.e., the map $N: M \to S^{n+1}$ is an immersion, then r-minimality of (X, M) is equivalent to $(n - r)$-minimality of (N, M). Indeed, nondegeneracy of the gauss map is equivalent to the nonvanishing of S_n. In this situation it is easy to show that the principal curvatures for (N, M) are the reciprocals of those for (X, M), and the rest is easy.

We conclude this section by considering two variational problems for surfaces in S^3. The first problem is to minimize $\int_M (S_1^2 - 4S_2)dV$. It is a simple calculation that $S_1^2 - 4S_2 = (k_1 - k_2)^2 \geq 0$ with equality only at umbilics. Thus this integral measures to what extent the surface fails to be a subsphere of S^3. One computes that the Euler-Lagrange equation is $2\Delta S_1 = -S_1(S_1^2 - 4S_2)$ where Δ is the Laplace-Beltrami operator. The following theorem is thus trivial to prove.

Theorem F. *A closed extremal for the problem* $\delta \int (S_1^2 - 4S_2)dV = 0$ *in* S^3 *whose first mean curvature has constant sign must be either a subsphere or a closed minimal surface.*

The second problem is that of minimizing $\int_M (S_1^2 - 2S_2)dV$. The integrand is clearly $k_1^2 + k_2^2$, so the integral measures the extent to which M deviates from being totally geodesic. The Euler-Lagrange equation is $2\Delta S_1 = -S_1(S_1^2 - 4S_2 + 2)$. Since $S_1^2 - 4S_2 \geq 0$, we see

Theorem G. *The only closed extremals of the variational problem* $\delta \int_M (S_1^2 - 2S_2)dV = 0$ *in* S^3 *whose first mean curvatures never change sign are the closed minimal surfaces.*

4. Applications to integral formulas in E^{n+1}

We have computed the variation of certain integrals. It is reasonable to search for those integrands for which $\int_M f(S_1, \cdots, S_n, P, Q)dV$ remains unchanged under any deformation. For example, $f = S_n$ is such a function. (When n is even, this invariance is part of the Gauss-Bonnet-Chern theorem.) We view such deformation invariant integrals as yielding integral formulas. Of course the formula need not be exactly the same for two immersions which cannot be joined by a deformation through immersions.

It is clear that (5) gives us the condition for deformation invariant integrals, at least all those involving only $S_1, \cdots, S_n, P, Q$. (We choose these quantities $S_1, \cdots, Q$ since many of the known integral formulas involve only them.) That is, a necessary and sufficient condition for $\int_M f(S_1, \cdots, Q)dV$ to be invariant under deformation is that

$$\begin{aligned} &-S_1 f(S_1, \cdots, Q) + \sum_{r=1}^{n} (S_1 S_r - (r+1)S_{r+1})D_r f(S_1, \cdots, Q) \\ (12) \qquad &+ \sum_{r=1}^{n} (D_r f(S_1, \cdots, Q))_{,ij} T_{r-1}{}^{ij} + (PS_1 + n + 1)D_{n+1} f(S_1, \cdots, Q) \\ &+ (D_{n+1} f(S_1, \cdots, Q))_{,j} Q_{,}{}^{j} + PD_{n+2} f(S_1, \cdots, Q) = 0 \end{aligned}$$

identically for all hypersurfaces. The study of (12) should give considerable insight into the nature of integral formulas. Up to now the development of interesting integral formulas has been a hit-or-miss affair.

We illustrate these ideas by giving a new demonstration of the well-known Minkowski-Hsiung integral formulas [8]. In terms of the modified curvatures they are $\int_M \left(P(r+1)S_{r+1} + (n-r)S_r\right)dV = 0$. Here M is closed, and $r = 0, 1, \cdots, n-1$. To prove these formulas, let $f(S_r, S_{r+1}, P) = (r+1)PS_{r+1} + (n-r)S_r$. One easily checks that for this choice of f, (12) becomes

$$(13) \quad P_{,ij}T_r{}^{ij} + (S_1 S_{r+1} - (r+2)S_{r+2})P + (r+1)S_{r+1} + S_{r+1,k}Q_{,}{}^{k} = 0 \,.$$

The fact that (13) is true for all hypersurfaces follows from the following easily derived formula:

$$(14) \qquad P_{,ij} = -b_{ij} - b_i{}^k{}_{,j} Q_{,k} - Pb^{(2)}{}_{ij} \,.$$

Indeed, (13) follows from (14) by multiplying (14) by $T_r{}^{ij}$, summing over i and j and doing the kind of manipulations we have been doing all along. Thus $\int_M ((r+1)PS_{r+1} + (n-r)S_r)dV$ is invariant under deformation. However under the deformation $X_t = tX$ it is clear that $\int_M ((r+1)PS_{r+1} + (n-r)S_r)dV$

goes into $t^{n-r}\int_M ((r+1)PS_{r+1}+(n-r)S_r)dV$. Thus, when $r<n$ the only way the integral can be invariant is if it vanishes. When $r=n$, our argument breaks down, and indeed the Gauss-Bonnet integral need not vanish.

5. Observations and questions

A. If M is a hypersurface in E^{n+1}, then the quantities $S_0=1, S_2, \cdots, S_{2q}, \cdots, 2q \leq n$, are intrinsic, i.e., definable in terms of the metric alone. In fact they are, up to constant factors, the intrinsic curvatures of Weyl [15]. In like manner the Newton tensors T_{2q}, $2p \leq n$, can be defined in terms of the metric. In fact, they are, up to constant factors, the generalized Einstein tensors defined by Lovelock in [9]. It is proved in [9] that the generalized Einstein tensors span the space of all symmetric divergence-free tensors of type (1, 1) which are concomittants of the metric tensor and its first two derivatives.

In [12] we consider the problem $\delta\int_M S_{2q}dV=0$ for submanifolds of arbitrary codimension. We prove that the Euler-Lagrange equation is $\boldsymbol{S}_{2q+1}=0$, where $(2q+1)\boldsymbol{S}_{2q+1}=T_{2q i}{}^{j}\boldsymbol{b}_j{}^{i}$, $(\boldsymbol{b}_j{}^{i})$ being the matrix of the vector valued second fundamental form.

The key to Theorem C was the equation $X_{,ij}=b_{ij}N$. The analogous equation when the codimension is arbitrary is $X_{,ij}=\boldsymbol{b}_{ij}$. Using the proof of Theorem C we quickly get the following generalization.

Theorem C*. *A necessary and sufficient condition for a submanifold M^n of E^{n+q} to be r-minimal, where $r=2l\leq n$, is that each component of the position vector satisfy the equation $T_r{}^{ij}\phi_{,ij}=0$.*

Example. It is easy to show that the Hsiang example $SO(3)/(z_2+z_2)\subset S^4\subset E^5$, is 2-minimal but not 0-minimal in E^5.

Just as Theorem C generalized to Theorem C*, so can Theorem E be generalized to a Theorem E^*. The statement is left to the reader.

B. The reason we have not considered general problems of the form $\delta\int_M f(S_2, S_4, \cdots)dV=0$ for submanifolds of arbitrary codimension is that our methods just do not seem to extend to that case. The major stumbling block appears to be the equation $T_rB=BT_r$, which may not be true.

C. Given a Riemannian manifold M^n and an even integer r, does there exist an r-minimal isometric immersion into some Euclidean space? Theorem C* tells us to study the equation $T_r{}^{ij}\phi_{,ij}=0$. If M is nice enough, for example, if M is a symmetric space, one may be able to say something.

D. When r is even, the equation $T_r{}^{ij}\phi_{,ij}=0$, which makes intrinsic sense on any Riemannian manifold, is clearly a generalization of the Laplace equation $\Delta\phi=0$. Like the Laplace equation, ours can be put in divergence form

$(T_r{}^{ij}\phi_{,i})_{,j} = 0$, and it is the Euler-Lagrange equation for a Dirichlet-type problem, namely, the problem $\delta \int_M (T_r{}^{ij}\phi_{,i}\phi_{,j})dV = 0$.

E. Under what conditions is T_r positive definite? Will this be the case if all sectional curvatures are positive? Since for hypersurfaces the T_r all have the same eigenvectors, it is of interest to see if the various T_r commute with each other in the general case.

F. Is there any deformation invariant integral $\int_M f(S_1, \cdots, S_n)dV$, involving only the modified mean curvatures, besides $f = S_n$?

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